Research Article https://doi.org/10.12973/ijem.10.3.479

International Journal of Educational Methodology
Volume 10, Issue 3, 479 - 494.

ISSN: 2469-9632
http://www.ijem.com/

Articulation Between a Technological Model and an Educational Model to Deepen the Reflection of Prospective Mathematics Teachers

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Received: March 26, 2024 ▪ Revised: June 3, 2024▪ Accepted: July 2, 2024

Abstract: This article is aimed at integrating the Technological Pedagogical Content Knowledge (TPACK) system with the Didactic Suitability Criteria (DSC) of the Didactic-Mathematical Knowledge and Competences (DMKC) system to improve the reflection of prospective mathematics teachers on online classes. Thirteen prospective teachers, divided into two subgroups, participated in a training cycle that addressed both models. Each participant used and created indicators of reflection of the assigned model to analyze an online class on functions, and subgroups exchanged reflections to examine the class from the other model's perspective. It was noted that the DMKC model allows for a broad analysis of the class but has limitations in assessing technology and the teacher's technological knowledge, while TPACK's emphasis is on technology and teacher knowledge but does not explicitly address mathematical interaction or affective aspects. It is concluded that combining the TPACK model and the DSC of the DMKC model can generate more complete tools to reflect on online math classes and consequently allow for a comprehensive evaluation that covers both the mathematical content and the technological and pedagogical skills of the teacher.

Keywords: *Didactic suitability criteria, DMKC, teacher reflection, TPACK, training of prospective mathematics teachers.*

To cite this article: Morales-López, Y., Breda, A., & Font, V. (2024). Articulation between a technological model and an educational model to deepen the reflection of prospective mathematics teachers. *International Journal of Educational Methodology*, *10*(3), 479-494. https://doi.org/10.12973/ijem.10.3.479

Introduction

In Mathematics Education, several theoretical models characterize the essential knowledge and skills that math teachers need to effectively perform their teaching tasks (Chapman, 2014). These theoretical models are designed for face-to-face math teaching that incorporates the use of technology because, at the moment, technology is considered a pillar of the development of society.

The importance of technology has also led to the emergence of knowledge models that do not focus on teaching a discipline, but are rather designed to attend to the problem of the knowledge needed by the teacher to incorporate technologies, particularly the Technological Pedagogical Content Knowledge model (TPACK) (Mishra & Koehler, 2006, 2008). This system or model is developed based on Shulman's (1986, 1987) papers, mainly on the notion of pedagogical content knowledge (PCK). Mishra and Koehler (2008) point out that TPACK is a model with knowledge domains and subdomains that tries to address the intricate connections among content, pedagogy, and technology knowledge. For insights into the implementation of this model in teacher training, consult the literature review by Gonçalves and Richit (2023).

As a result of the presence of technology in the classroom, interest in the knowledge and competences necessary to teach math with technological resources has increased. Literature from different parts of the world (e.g., Huang & Zbiek, 2017; McCulloch et al., 2021; Morales-López, 2017; Potari & da Ponte, 2017) shows deficiencies in mathematical, didactic, didactic-mathematical, and technological knowledge and competences in prospective secondary mathematics teachers (PSMT). This becomes more relevant because the knowledge and competences developed by PSMTs are directly linked to the knowledge and competences that will be required from and must be developed by their students in basic education (J. D. Godino et al., 2017; Kim & Albert, 2015).

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In addition, the COVID-19 pandemic created new problems and showed that the knowledge and competences necessary for prospective mathematics teachers are still far from being fully understood, especially those concerning the use of technology. Along the same lines, Engelbrecht et al. (2023), Atweh et al. (2023), Villarreal et al. (2023), and Ávila (2023) point out the urgency to make comparisons and contrasts to consider what was learned from the pandemic in terms of mathematics education and improve the present and forthcoming scenario, especially what is linked to the use of technology (Calle et al., 2021; Villa-Ochoa et al., 2023). In this context, additional research is needed on teaching and learning mathematics in both online and hybrid modalities, that is, when technology is not only a resource but a means. This need entails developing knowledge models designed for face-to-face mathematics teaching that can be adapted to online and hybrid teaching.

One of the knowledge and competences models initially developed with face-to-face teaching in mind is the Didactic-Mathematical Knowledge and Competences model (DMKC). As a result of the Onto-Semiotic Approach (OSA) (J. D. Godino et al., 2007), a framework outlining the knowledge and competences of mathematics teachers was created, entitled DMKC (Pino-Fan et al., 2023). Within DMKC, one of the key competences of mathematics teachers is didactic analysis and intervention, and one of its subcompetences is the assessment and analysis of Didactic Suitability (DS), which allows for, among other things, the judgment of what happens in the classroom. The competence analysis and assessment of DS is developed by learning and using the Didactic Suitability Criteria (DSC) tool, which is a construct taught in some mathematics teacher training programs.

This article is aimed at advancing the articulation between the TPACK model and the DSC of the DMKC model to offer future mathematics teachers a tool to improve their reflection on online mathematics classes. The context for this articulation is the reflections by prospective teachers from a Costa Rican university when asked to analyze situations that occurred in an online mathematics class from two different perspectives (TPACK and DMKC), creating and using their own indicators, if necessary. One of the PSMT groups developed their indicators from the TPACK model and then conducted an analysis of the class, while the other conducted a similar activity creating and using indicators but conducting the analysis from the DMKC model. Both groups were provided with an insufficient tool since they had to reflect on a math class in which technology had a fundamental role. They had to use either the TPACK model, which was theoretically designed to analyze technology-related aspects in detail but does not reflect on a class as a whole, or the DMKC model, specifically the DSC tool, which allows a comprehensive reflection on the math class but does not focus on technology. Given the inadequacy of the tool provided, participants had to look for new tools and considerations for professional reflection on the smart and timely incorporation of technology in math classes. The extensions of each model created by the participants are the basis for the articulation of both models.

The main motivation for this research lies in the need to adapt and expand established didactic models to respond to contemporary challenges in education. Although DMKC is a very robust model for didactic research and analysis, its original conception was focused on face-to-face classes, which limits its applicability in the context of activities derived from online teaching, which has become highly relevant in recent years due to various circumstances, such as the COVID-19 pandemic. On the other hand, TPACK has become a highly cited framework in international literature. This model offers a valuable framework for understanding and assessing teacher knowledge in the integration of technologies in the educational process, which is crucial in a world where digital tools are becoming more and more relevant in teaching. Integrating DMKC and TPACK may open a new research agenda to strengthen mathematics education in non-traditional contexts. In this regard, the bibliometric study by Chacón-Rivadeneira et al. (in press) on teachers' didactic-mathematical knowledge and competences, which is based on research using TPACK or DMKC as theoretical references, concludes that an important advance is the integration of TPACK with DMKC to develop and assess the competences needed to incorporate technology in mathematics instruction.

The following sections provide an overview of the key educational frameworks and models that are crucial to justify such possible integration. These include the DMKC model from the OSA framework, the TPACK model, and the implementation of the network theory to link the two models within this specific context. Each section explores in depth the fundamental principles and foundational components of these models.

Literature Review

DMKC Model

Within the OSA framework (J. D. Godino et al., 2007), a theoretical model of didactic-mathematical knowledge (DMK) (J. D. Godino et al., 2009; Pino-Fan et al., 2015, 2018) of the mathematics teacher was first developed. One aspect of the model's development involved establishing a connection between the concept of knowledge and the concept of competence.

Moreover, within the OSA framework, studies have explored the competences of mathematics teachers, revealing the need for a model of teacher knowledge to assess and enhance their competences. These parallel lines of inquiry converged, giving rise to the DMKC model (Breda et al., 2017; Font Moll et al., 2015; Giacomone et al., 2018; Godino et al., 2017; Pino-Fan et al., 2023; Pochulu et al., 2016; Seckel & Font, 2015).

In this model, the essential competences for mathematics teachers include mathematical proficiency, didactic analysis, and intervention skills. At the heart of the latter lies the capacity to create, implement, and evaluate learning sequences (Breda et al., 2017), utilizing didactic analysis methods and quality standards. This is aimed at setting up cycles for planning, implementation, and evaluation, as well as suggesting areas for enhancement. This paper places particular emphasis on examining the sub-competency related to evaluating the Didactic Suitability (DS) of instructional processes.

The initial analysis step delves into the mathematical practices enacted within a mathematical instructional session. The subsequent level scrutinizes the mathematical elements and procedures that are central to these practices, alongside their resulting outcomes. The third tier of didactic analysis is primarily related to delineating interaction patterns (Leguizamón Romero, 2017), didactic configurations, and their sequential manifestation in didactic trajectories. The fourth level of analysis concentrates on the criteria governing the instructional process (Godino et al., 2009). Finally, the fifth level builds upon the prior four levels of analysis, aiming to pinpoint potential improvements for the instructional process in subsequent implementations (Breda et al., 2018; Font et al., 2010; Godino, 2013; Malet et al., 2021). All these tools seek a didactic analysis that allows for a reflective design, implementation, evaluation, and assessment of instructional processes (Godino, 2022). Enhancing competence in didactic analysis and intervention empowers educators to perform the various types of didactic analyses advocated by the OSA framework. Consequently, training programs aimed at teaching and learning these analytical methods contribute to the cultivation of this competence and the acquisition of the necessary pedagogical knowledge outlined in the DMKC model.

A primary theoretical tool used within the DMKC model revolves around the concept of didactic suitability (DS), which is defined as:

> “The degree to which such process (or a part of it) meets certain characteristics that qualify it as optimal or adequate to achieve the adaptation between the students’ personal meanings (learning) and the intended or implemented institutional meanings (teaching), considering the circumstances and available resources (environment)” (Godino et al., 2023, p. 7).

Suitability is described into six criteria within the DMKC model: “the epistemic suitability criterion”, which evaluates the quality of the mathematics being taught; the “cognitive suitability criterion”, which assesses the alignment between the teacher's objectives and students' existing knowledge both before and after the instructional process; “the interactional suitability criterion”, which gauges the effectiveness of interactions in addressing student inquiries and challenges; “the mediation suitability criterion”, which evaluates the appropriateness of resources and time allocation during instruction; the “affective suitability criterion”, which measures student engagement and motivation throughout the instructional process; and the “ecological suitability criterion”, which examines the alignment of the instructional process with the educational institution's mission, curriculum guidelines, and broader socio-professional context. To implement these criteria for the analysis and evaluation of instructional processes, each suitability type is associated with a system of characteristics and components. Table 1 outlines the specifics of DS characteristics. A comprehensive overview is available in Breda et al. (2017).

Table 1. Characteristics and Components of DS (Breda & Lima, 2016, pp. 80-83)

Characteristics	Components
Epistemic (ES)	“Errors, ambiguities, diversity of processes, representation”
Cognitive (CS)	“Previous knowledge, adaptations of the curriculum to the individual’s different needs, learning, high cognitive demand”
Interactional (IS)	“Teacher-student interaction, interaction between learners, autonomy, formative evaluation”
Mediational (MS)	“Material resources, number of students, scheduling, classroom conditions, time”
Affective (AS)	“Interests and needs, attitudes, emotions”
Ecological (ECS)	“Adaptation to the curriculum, intra/interdisciplinary connections, social-professional practicality, didactic innovation”

The development of DSC components and features considered contemporary directions in mathematics education, principles from the National Council of Teachers of Mathematics, and findings from well-regarded research endeavors in the mathematics education field, which enjoy broad acceptance within the academic community (Breda et al., 2018). For instance, in ES the following is reflected a result of the research in mathematical education: mathematical objects are products of the practices, which entails their complexity. The component "representation of the complexity of the notion to be taught" is derived from this result, and its objective is to contemplate the mathematical intricacy of the object to be taught to guide teachers in the design and redesign of learning sequences.

The DS concept has made a substantial impact on teacher education programs worldwide. This influence underscores the integration of DSC in various research initiatives concerning mathematics teacher training, as this construct serves as a central component of a dedicated training tool aimed at facilitating educators' reflections on their teaching practices. (e.g., Calle et al., 2023; Esqué de los Ojos & Breda, 2021; Giacomone et al., 2018; Hummes et al., 2023; Morales Maure et al., 2019).

TPACK Model

The TPACK model is based on the work by Shulman (1986, 1987) on teacher knowledge and has three main domains: 1) "content knowledge" (CK), which refers to the knowledge of the subject, in this particular case mathematics; 2) "pedagogical knowledge" (PK), which refers to understand how to educate; and 3) "technological knowledge" (TK), which is knowledge about technological resources and skills to handle both hardware and software and different types of devices. The following subdomains derive from the three domains mentioned above: 4) the subdomain pedagogical content knowledge (PCK) corresponds directly to the knowledge a teacher must have in a discipline to be able to teach it and directly deals with Shulman's (1986, 1987) work; 5) the technological content knowledge (TCK) studies the correspondence between content and technology, both potentially and effectively. In the realm of mathematics, this entails, for instance, the way of conceiving mathematical objects with technology (representations, definitions, etc.) mainly oriented to the development of mathematics; 6) technological pedagogical knowledge (TPK) is concerned with how technology is conceived, learning, and teaching in general; here you can find learning theories such as Siemens' connectivism (2005); and finally, 7) the subdomain technological pedagogical content knowledge (TPACK) represents the core of the type of knowledge that a teacher must have to incorporate technology in an educational process intelligently.

It should be noted that several alternative theoretical models in mathematics have derived from the TPACK model, such as a) STAMPK (Getenet et al., 2015); b) TPMK (Koh, 2019); and c) KTMT (Rocha, 2020). However, there is really still very little scientific literature to determine how robust they are and, if at some point, they will be consolidated.

Up to this point, it can be stated that there are some reasonable similarities between DSC and TPACK. From a mainly theoretical perspective, a significant relationship is observed between SD's epistemic and cognitive criteria and the following TPACK domains: mathematical content knowledge, pedagogical knowledge, and pedagogical content knowledge. Both models search for deep knowledge about mathematical objects and their various representations, as well as the cognitive processes that are activated during learning. In addition, both models emphasize the importance of the appropriate use of students' prior knowledge and the construction of new concepts through effective educational scaffolding. This common search is reflected in the attention that both models pay to how teachers can facilitate learning through strategies that allow concepts to emerge from instructional practices.

It is important to note that these theoretical connections only become evident when contrasting the two models. TPACK alone is not a particularly detailed tool to address the various mathematical and didactical-mathematical objects and concepts with the specificity offered by SD. However, by integrating the SD criteria with TPACK knowledge domains, a more holistic and enriched understanding of mathematics teaching and learning seems to be obtained. It is safe to assume that the potential integration of these models may allow for the development of more comprehensive approaches to the didactic analysis of situations with technology.

Articulation between the TPACK and the DMKC Models

Different methods can be used to address the research problem regarding theoretical or network articulation. There are different stages in this networking process that can range from ignoring other frameworks to completely merging two or more theories (see Figure 1).

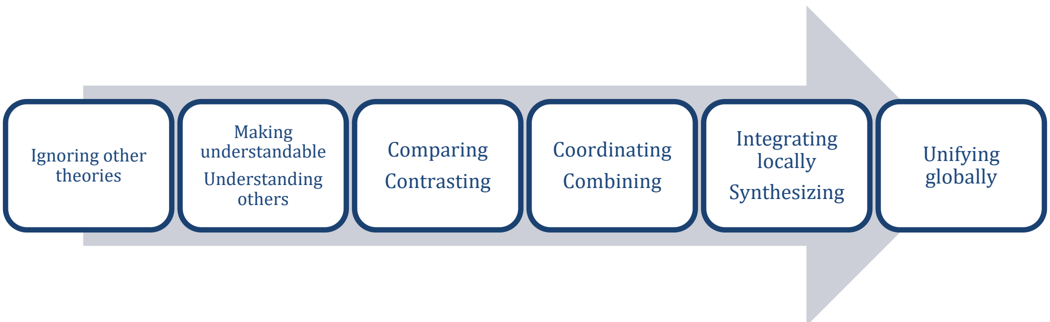

Figure 1. Degrees of Integration to Connect Theoretical Approaches (Source: Designed by Authors Based on Prediger et al., 2008, p. 170)

Within the OSA framework, several research projects have been conducted that seek articulation between different theoretical models.

A meta-analytical study conducted by Breda et al. (2021) finds OSA articulation with the following notions and theoretical frameworks: the notion of meaning by Louis Hjelmslev, Charles Sanders Peirce, and Ludwing Wittgenstein, as well as by Friedrich Ludwig Gottlob Frege, Gérard Vergnaud, and Horst Steinbring (Godino et al., 2022); Extended Theory of Mathematical Connections (Rodríguez-Nieto, Font Moll et al., 2022; Rodríguez-Nieto, Rodríguez-Vásquez et al., 2021); Luis Radford's theory of objectification (Godino et al., 2020); Ed Dubinsky's Action, Process, Object, Schema (APOS)

Theory (Font Moll et al., 2016); Raymond Duval's theory of registers of semiotic representations (Godino et al., 2016); Guy Brousseau's theory of didactic situations, Yves Chevallard's anthropological theory of the didactic, and Gérard Vergnaud's theory of conceptual fields (Godino et al., 2006); Theory of Instrumental Genesis (Drijvers et al., 2013); Ethnomathematics (Oliveras & Godino, 2015); Modelling from a cognitive perspective (Ledezma et al., 2023). In short, OSA represents a teacher knowledge model that integrates and expands the development and advances of various mathematics teacher knowledge models, especially the models by Lee Shulman et al. and Deborah Ball et al. (J. D. Godino et al., 2017; Pino-Fan & Godino, 2015). Finally, Chacón-Rivadeneira et al. (in press) conducted a bibliometric study between TPACK and DMKC. Despite the theoretical similarities and points of convergence mentioned above, literature shows that, up to now, there has been no systematic effort to integrate or compare them empirically. This may represent a significant gap and an important opportunity, given the potential of combining these frameworks to enrich the mathematics teaching practice. Although the creation of a new theoretical framework is not intended, integrating TPACK with DMKC could provide a particularly useful tool that addresses both the challenges of traditional mathematics teaching as well as those arising in the context of technological and online education. This could lead to a better understanding of how teachers can combine their mathematical, didactic-mathematical, and technological knowledge to teach mathematics in different educational contexts.

The main objective of this research study was to develop a novel coordinated and combined strategy between TPACK and DMKC, by selecting two groups of participants and providing each group with one of the two models to perform the same task (reflect on an online mathematics class). The model was insufficient for the task so students would need to develop the model and then study if these developments fell within the constructs of the other model.

Methodology

Research Design

This research was proposed from a qualitative approach with an exploratory perspective, adapting some aspects of the networking methodology. Data was collected through questionnaires called activities in the latter half of 2022.

Sample and Data Collection

The study included 13 students of the "Bachelor's and Licentiate Programs in Mathematics Education at the National University of Costa Rica" enrolled in the course Educational Research Seminar 2, which corresponds to the tenth cycle of this program (five and a half years).

All students digitally signed an informed consent form to participate in the research. This process of obtaining consent was essential to ensure the ethics and transparency of the study. Informed consent not only ensured that students fully understood the purpose, procedures and potential risks and benefits of the research, but also respected their autonomy and right to decide about their participation. Although the university where the research was conducted does not have a mandatory structure to conduct this type of research, ethical practices suggested in international literature were followed. In particular, confidentiality was maintained by changing personal data in the databases, and internal codes were established by the researchers to code this information. Given that the activities were conducted online, this coding was particularly relevant, as it allowed for efficient tracking and strict confidentiality.

Videorecording

Attendees were prompted to examine a 120-minute video footage capturing a mathematics lesson on functions. This session was conducted virtually via Zoom by three high school instructors.

The class addressed basic notions related to functions (such as the definition, characteristics, and representations) and was organized as follows: a) an initial problem was posed based on the launching of a ball and its trajectory, the maximum and minimum height of the ball on the path, maximum and minimum time it traveled until it hit the ground, intervals of time when the ball ascended and descended, and particular cases of height value or value of travel time; b) individual projects of students on the initial problem solving questions posed by teachers; c) online discussion and exercises with the Nearpod application and an explanation of the solutions of the exercises with the Awwapp virtual whiteboard; d) study of the notion of function from the representation of different graphs (not only with the representation of the quadratic function graph) with the use of GeoGebra and sliders; e) reinforcement of the concepts learned with questions in Nearpod (competency and gamification questions); and finally f) an activity to link the function content with a real context (the COVID-19 context). A graph with COVID-19 active cases published in one of the newspapers in Costa Rica was presented here; this graph was linked to the notions learned (increase in COVID-19 cases per day, peak cases, daily ratio, and number of cases as examples of pre-image and image, the concept domain, and other notions worked on in the initial activity).

Research Protocol

During the research period, eight sessions were held, each including five hours of classes, six hours of independent work to do the assigned tasks and readings on the subject of each session, and two hours to answer specific questions. The first two sessions consisted of addressing general elements on the concept of reflection, international trends on the vision of mathematics didactics, and the system of organization of knowledge proposed by Shulman (1986, 1987) and some of its derivatives in mathematics education. Specifically, in the second session, the total group of 13 teachers was divided (6 that were to study the DMKC model in the following sessions and 7 that were going to study TPACK). In this way, in the second session, each group received guidance, and the nature and basic structure of the corresponding model were studied.

In session 3, some of the tools that make the assigned model operational were explained to each group to analyze and reflect on the video recording of the online class. The concept of DS and its criteria were described in detail to the group that was assigned the DMKC model. Particularly, the DSC construct found in Breda and Lima (2016) was taught to this group. The TPACK group studied the indicators developed by Morales-López et al. (2021) and Schmidt et al. (2009). Subsequently, the first activity (Activity #1) was assigned, which consisted of observing and describing an online class on functions. To do this, they had to organize the observation's description based on the use of indicators, which could be any of the ones already studied or the ones created by them if they considered that the those studied were insufficient to organize what was observed.

Session 4 was intended as a space for students to make individual progress on Activity #1. In session 5, the second activity (Activity #2) was assigned. This activity consisted of an individual written reflection that considered descriptive, explanatory, and evaluative elements of the online class on functions based on the rubric created in Activity #1 by each student and the model they previously studied. Session 6 was intended for students to move forward with Activity #2.

Session 7 consisted of explaining and starting Activity #3. For this activity, each participant of the DMKC group was given a reflection created by a student of the TPACK group, while each participant of the TPACK group was given a written reflection made by a student from the DMKC group and was asked to compare the reflection they made using their assigned theoretical model with the one made by the other participant using the other theoretical model. This organization was possible as only 10 prospective teachers participated, paring five from each group. Session 8 was dedicated to working on Activity #3.

Although all the reflections were studied by the researchers, for this paper, only the one that researchers considered more elaborate was selected in each group; the main criteria used by the researchers were clarity of writing, completeness of ideas, representativeness of individual comments in relation to the group. For purposes of this research, the PSMT in the DMKC group will be referred to as Charles, and the other TPACK participant will be referred to as Timothy. In order to provide greater clarity and understanding of the activities carried out, the main questions and instructions addressed to the participants have been included in appendices 1, 2, and 3. These appendices are intended to provide a more complete context and facilitate an understanding of the procedures and actions conducted during the training cycle. Appendix 4 shows a description of the objectives and activities developed to ensure a consistent understanding of the topics developed.

The following section shows a brief description of what was done by the two selected teachers in training in Activities #1 and #2. These two activities are directly integrated into Activity #3, as they were first used to create the initial rubrics and reflections and were later the basis where each student could compare the results of Activity #3 with those of their peers. Regarding the technique and strategy to analyze and interpret information, Content Analysis (Bardin, 1996) was used to obtain indicators for the inference of knowledge related to contexts in the following steps a) pre-analysis (information is organized and possible conjectures or interpretations are established), b) material exploration (indicators are defined, codified, and listed), and c) data processing and interpretation (data is converted into meaningful and valid information).

Results

Activity #1

Regarding the first activity, Charles developed a battery with 26 analysis indicators, of which, according to him, five were extracted verbatim from Breda and Lima (2016), nine were adapted from indicators proposed by these authors, and twelve were created by him. The indicators created by Charles are linked to representations of mathematical objects (in this case, functions), prior knowledge necessary for the study of functions, variety of explanations, difficulties when learning functions, promotion of the activity among students, time management, use of technological tools, contextualized tasks and adaptation of the curriculum to the technological environment. It should be noted that, although he considers that his indicators are his own, these topics were already fully considered in the indicators of the DMKC didactic suitability criteria. For this same activity, Timothy established a battery with 21 analysis indicators, of which 17 were identical indicators or adaptations from Morales-López et al. (2021) and Schmidt et al. (2009), and the remaining four were created by him. These indicators were related to the adaptation to the needs of remote presence, use of technological resources, various teaching approaches, and use of specific software for teaching functions.

Activity #2

In Activity #2, each participant wrote a reflection on the class using their corresponding models. In general, both participants share the fact that they used, as their reflection organizers, the tools of their assigned model, that is, domains and subdomains for TPACK, as well as DSC criteria, components, and indicators for DMKC. Charles' reflection integrated and balanced both descriptions and assessments. Timothy, in turn, presented two completely separate parts in his writing from the beginning (the first part was descriptive, while the second part combined description and assessment). There is an important difference in how they both used the indicators they developed. Timothy used indicators as organizational elements, trying to explicitly include every one of them in his writing (even making them the initial sentence of each paragraph), which helped justify the presence or absence of each indicator. Instead, Charles first wrote his reflection and then, in a way, did a content analysis to justify that his comment could be re-explained using the indicators of the model, which gave his writing more freedom and fluency. Table 2 below provides an example of the type of indicators that both participants were using for their reflection.

Table 2. Example of Indicators Created by Each Participant (Source: Derived from This Research)

Timothy (TPACK)			**Charles (DMKC)**		
Domain and subdomain	Developed by authors	Exact copy or adaptation (indicator)	Criterion	Developed by authors	Exact copy or adaptation (indicator)
Pedagogical knowledge	Correctly adapts the way of teaching based on the student's educational needs at a given moment (in-person, online).	Able to assess students' learning in different ways.	Mediational	Adequately uses technological tools to assist the teaching and learning of the contents presented.	The distribution of time for each of the contents is adequate according to the level of complexity and relevance.

Findings

Activity #3

In Activity #3, each PSMT was given a reflection created by a peer from the other group. Consequently, Timothy, who had created a TPACK-based reflection, analyzed a DMKC-based reflection, while Charles, who created a DMKC-based reflection in Activity #2, was now analyzing a TPACK-based reflection created by another peer. It should be noted that Charles was not given Timothy's reflection or vice versa.

Regarding the indicators present in the reflection they read and also included in their own reflection, Charles pointed out that his colleague's reflection highlighted aspects that he had also considered, but that were associated with different indicators due to the model used. For example, Charles indicated that having one of the teachers who taught the class clarify technological questions was not a matter of support, but rather a matter of lack of prior knowledge and planning in case of possible student problems. Similarly, Charles indicated that the reflection he read from his partner highlights the increased cognitive demands and the difficulty of the activities, but he did not consider this to be the case. Charles considered that the reflection that he read emphasized the planning that could have been made for the class and the description of the stages, while he considered this one more element and gave no special attention to it.

Charles considered that, in the reflection he read, there were inferred indicators with little justification from aspects that he did not take into consideration, since, in his opinion, they were not very relevant:

> *I (Charles) think that saying that, by fixing a sound problem, teachers reflect part of their technological knowledge is incorrect. I feel that more than an example of software usage and knowledge, it was a connection issue.*

Finally, Charles indicated that there are certain inferred indicators not present in his reflection, that are important to consider:

> *I (Charles) think it is necessary to state that the author mentions the teachers' willingness to learn and put into practice new knowledge in technologies. This was not included in my reflection; however, I think it is important since, thanks to the experience gained over the years in this modality, there is a significant number of teachers who were not able to or did not show a desire to look for ways to diversify their tools, and rather limited themselves to reciting slides in their classes.*

In Timothy's case, when asked about aspects he did not consider but were considered by his peer in his reflection, he pointed out that he focused more on the "way of teaching using technology as a tool" and that is why he did not consider elements such as language (mathematical, colloquial, etc.). This is relevant because it shows that the battery he developed helped him focus more on the educational technology area, instead of focusing on elements of mathematics teaching.

Timothy notices that his partner's reflection has many indicators that he did not consider. For example, when reading the affective suitability analysis, he mentions that:

> *This was a characteristic that I (Timothy) did not take into account and that, now, reading this reflection, I consider it to be a very important component in the development of a class. However, from my point of view, when these indicators are detailed and analyzed, we must be very careful, since, in an online class, it is not possible to clearly perceive the emotions of the people on the other side of the screen; consequently, in this case, they must be based solely on the student's willingness to participate.*

Timothy not only recognizes the lack of indicators in key topics (such as affective criteria) but also tries to adapt it to an online math class.

When asked what elements he took into account in his reflection that were not considered by his colleague, Charles pointed out a lack of analysis of the interaction among people and stated that "*contextualizing the activities and studying them in depth was a vital point in my reflection (from a DMKC perspective), which does not seem to be observed in detail in the reflection analyzed in this document*".

Timothy, in turn, says the following about the reflection he read: "*my indicators were mainly aimed at the work done by the teacher and how he or she improved the teaching process using technology as a tool*", with which implicitly he recognizes that there are many elements that he did not take into account. However, Timothy does indicate that there are absences regarding technology in the other model; for example, "*it is not focused on the teacher's knowledge and mastery when using these platforms*". Finally, he mentions that, although there are many elements missing, his reflection also has an in-depth description and analysis.

Finally, both participants were asked to compare their reflections. Charles indicated that his peer's reflection was brief and lacked analysis and many indicators that he considered important (from his DMKC perspective). Timothy mentioned that, after reading the other reflection, he realized that there were many elements missing in his own reflection and that his emphasis was on technology. However, it is inferred that, from reading his partner's reflection, he identified aspects that were missing in his own reflection that can be used to enhance it. For example, by saying the following: "*I must recognize that, although this reflection [the one Timothy read] is written thinking of the Onto semiotic perspective, I can identify key ideas of this approach and even come up with some others to complement it with TPACK*", Timothy is offering evidence that this activity made him consider that a combination or coordination between both models could facilitate a deeper reflection.

Discussion

The general objective of this research was addressed by contrasting future teachers' arguments resulting from the study of events occurring in a virtual math class from the DMKC and the TPACK perspectives. The two models proposed and their complementary tools present advantages and disadvantages.

In the case of the DSC construct of DMKC, there is a tool that allows a broad analysis of everything that happens in a mathematics classroom. This tool allows us to understand the connections and interactions between people (teachers and students). This capability is fundamental in a framework such as this, as it provides a better understanding of multiple phenomena (Font et al., 2024; Pereira & Kaiber, 2022; Pino-Fan et al., 2023). However, since it is so broad, it presents limitations for an in-depth analysis of the specific use of technology in the mathematics classroom.

Meanwhile, within the TPACK framework, the divisions into domains and sub-domains, together with the additional indicators, offer an approach focused on the teacher's understanding of content, pedagogy, and technology, and several papers have already shown that prospective trainee teachers can have an adequate understanding of TPACK (Amidi et al., 2024). However, this approach may not have tools as explicit to analyze specific aspects such as mathematical activity, classroom interaction, and the affective-emotional elements involved. This raises a fundamental implication: certain indicators are more closely related to the TPACK domains and sub-domains than others. On the contrary, trainee teachers perceived the TPACK model as insufficient in comparison with the suitability criteria.

To overcome these limitations, it might be useful to complement DMKC's DSC model with additional TPACK tools that would focus specifically on the detailed analysis of the technology used in the classroom and on the development of the teacher's technological knowledge. Findings such as those of Hanifah et al. (2024) support this statement, for example, in the relationship between CK and the epistemic component.

This has implications for both models. How such synchronization can be expressed can range from the articulation of indicators that feed into more complete batteries, to the creation of new categories in both models. However, including new categories or dimensions is not trivial, as there is an epistemic and philosophical component behind each model. Aside from the intention of generating new frameworks, establishing articulations between the criteria can generate specific tools for various situations and a more complete and accurate view of mathematics education in technologically enriched environments.

Finally, regarding the links between DSC and the TPACK domains, a strong relationship is inferred between indicators of the Epistemic Suitability criterion and Content Knowledge. These strong connections between indicators also appear between indicators of the Cognitive Suitability criterion and Pedagogical Knowledge, and between indicators of the Mediational Suitability criterion and Technological Knowledge. The first two are consistent with the results obtained in Godino and Pino-Fan (2014) and Pino-Fan and Godino (2015) who studied approaches between OSA and the Pedagogical Content Knowledge proposed by Shulman (1986, 1987).

Conclusion

TPACK domains and subdomains and the batteries of the aforementioned indicators show limitations in analyzing aspects related to attitudes, beliefs, and emotions in the learning process (affective suitability criterion). In addition, there is a lack of vision of mathematical instruction to establish a relationship between the curriculum and the social, technological, and work environment, among others (ecological suitability criteria) (weak connections). Figure 2 shows a diagram with a description of the strong and weak connections determined.

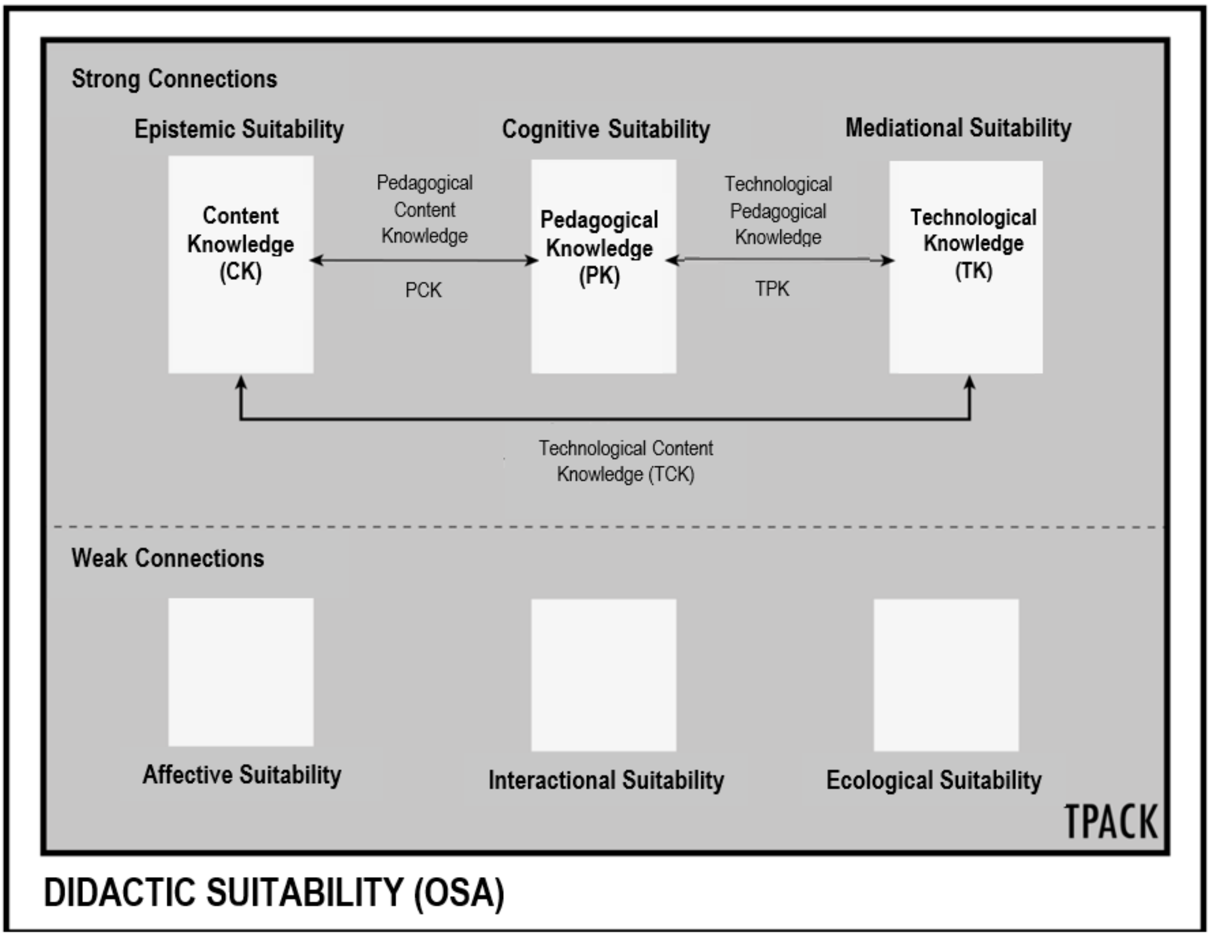

Figure 2. Strong and Weak Connections between Didactic Suitability Criteria and TPACK Domains and Subdomains (Source: Derived from This Research)

In conclusion, a coordinated combination between the TPACK and DMKC models sounds feasible and enriching for both models. This research, as well as the research by Morales-López et al. (2023) and Morales-López and Font (2024), offer strong evidence for this possibility.

Recommendations

Research should be applied to broader scenarios in which teachers create mathematical activities and tasks using technological resources and can justify them with their own DMKC-TPACK-derived tools. Other possible connections between other technological knowledge theories should be investigated since OSA has many methodological theoretical tools for didactic analysis. As shown in this paper, other connections can be explored with a similar methodology.

Limitations

A limitation that must be noted in this research is that the DSC construct taught to the group assigned DMKC were not adapted to functions. In other research projects, these indicators have been refined for functions. If these refined guidelines had been provided for functions, perhaps the indicators used by the participants would have been different since they would have had a specific guideline to conduct a more detailed analysis (Inglada Rodríguez et al., 2024).

Ethics Statements

The studies involving human participants were reviewed and approved by the Universitat de Barcelona. The participants provided their written informed consent to participate in this study.

Acknowledgments

This research is a direct result of the doctoral dissertation by Yuri Morales-López in the program "Didàctica de les Ciències, les Llengües, les Arts i les Humanitats" from Universitat de Barcelona, Spain. In addition, we would like to thank Universidad Nacional, Costa Rica for their support.

Conflict of Interest

No conflict of interest is declared by the authors.

Funding

This research was conducted as part of the project PID2021-127104NB-I00 funded by MCIN/AEI/10.13039/501100011033 and the European Regional Development Fund (ERDF) - "A way of making Europe". The Universidad Nacional in Costa Rica provided funding for both the translation and the Article Processing Charges (APC).

Authorship Contribution Statement

Morales-López: Conceptualization, data collection, drafting manuscript, data processing and evaluation, grant acquisition. Breda: Approval of final version. Font: Approval of final version.

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Appendixes

Appendix 1

General Guidelines for Research Activity #1:

1) Download and watch the video available in the virtual classroom; 2) Based on the domains discussed in the previous session (for OSA group 1: Epistemic, cognitive, affective, mediational, interactional, and ecological) (for TPACK group 2: T, P, K, PCK, TCK, TPK, and TPACK), develop indicators aimed at assessing a virtual mathematics class on the topic of functions (regarding the teacher's role and knowledge). You may use the indicators each domain has as a basis, adapt them, and create as many new ones as you consider necessary. Do not use documents from other groups. 3) Submit a table (battery) with the domains and indicators you developed for the assessment (remember they must be developed aiming at a particular type of class: virtual and with a specific topic: functions).

Appendix 2

General Guidelines for Research Activity #2:

Each student must write a 3-4-page reflection in MS Word on the virtual classroom video. This reflection should be based on the domains and indicators created in the individual rubric/battery. The reflection is based on two parameters: 1) Description and 2) Analysis. Writing is not tabular but in prose. You may use external sources, but the main purpose is to describe and analyze the situations proposed by your rubric. If there are elements that are not present in your set of indicators, you can modify it and attach the new version to the assignment.

Appendix 3

General Guidelines for Research Activity #3:

Each student must analyze the reflection written by a peer from the other group and write a free-style essay of at least three pages. The essay must compare their own reflection with that of their peer's and answer the following questions: 1) What interesting elements did your classmate highlight that you did not consider; 2) What elements did you consider in your reflection that this person did not consider; 3) Regardless of the model and based on the document, do you consider that there are both descriptive and analytical elements?; 4) Explain how you would compare your reflection with the reflection this person wrote (level of description, depth, among others).

Appendix 4

Table A1. Objectives, Activities, and Materials of Each Session (Source: Derived from This Research)

Session	Objective	Activities and materials
First	To provide students with a comprehensive overview of the main international trends in mathematics education. This includes exploring current and emerging pedagogical approaches and globally adopted practices.	To achieve this objective, we began by explaining national trends reflected in the national curriculum, providing students with a solid grounding in the local educational context. From there, international trends were explored in depth by studying and analyzing readings by globally recognized organizations in the field of mathematics education, such as the National Council of Teachers of Mathematics (NCTM) and the International Commission on Mathematical Instruction (ICMI), among others.
Second	To introduce students to the subject of teaching knowledge as a fundamental pillar of contemporary educational research.	In this activity, students are expected to understand the importance of specialized teacher knowledge, which includes not only subject proficiency but also pedagogical skills and an understanding of school dynamics, through the study of the papers by Shulman (1986, 1987).
Third	To explain to the students two different models that can be used to carry out an analysis of classroom events.	The group was divided into two subgroups, and each subgroup was assigned to study a specific model of event analysis in the classroom. This division allows each subgroup to focus in depth on their assigned model, facilitating a more detailed and specialized understanding. Materials used for this study include key readings: Breda and Lima (2016), Morales-López et al. (2021), and Schmidt et al. (2009).

Table A1. Continued

Session	Objective	Activities and materials
Fourth	To generate a space for individual work in order to reflect on the materials studied at the moment to foster a deeper and more personal understanding of the concepts addressed.	The focus was on Activity #1, using as a basis the materials reviewed in sessions 1, 2, and 3. This activity is designed to consolidate the products of Activity #1.
Fifth and sixth	To explain and substantiate the elements of analysis and reflection for Activity #2.	The activity consisted of an individual reflection on the class observed, using a rubric developed specifically for this purpose. The rubric they developed provides them with a guideline for their analysis. The work primarily focuses on Activity #2, using the materials reviewed in previous sessions as a basis. This activity is designed to consolidate the products of Activity #2.
Seventh and eighth	To generate a space for individual work in order to create a reflection.	The activity consists of reading and analyzing a reflection written by a classmate from another group, which facilitates the comparison and contrast of different perspectives and approaches in relation to the same class observed. This exercise, called Activity #3, provides input for the proposed research, as students can identify common patterns, similarities, and differences, as well as possible topics of interest to explore how to incorporate technology in the classroom.